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Exploring Entanglement Transitions in the 4-Level Qudit Projective Transverse Field Ising Model

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Quantum Error Correction

- Quantum computers are powerful, but present physical challenges – decoherence, noise
- Large-scale implementations require quantum error correction
- Apply projective measurements as syndrome measurement to counteract error
- Transitions allow us to study competing behavior between projective measurements and long-range entanglement to determnine feasible rate of measurement

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$\mathbb{Z}_2$ Transverse Field Ising Model

Hamiltonian H given as

$$H = -J\left(\sum_{\{i,j\}} Z_i Z_j + g \sum_i X_i\right)$$

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

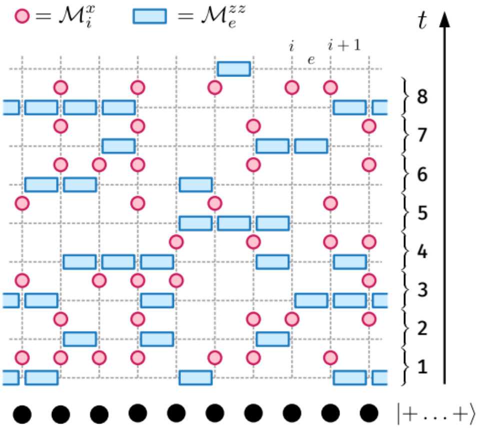
- Nearest neighbor interactions described by alignment of Z spins
- Influence from external magnetic field in X direction
- Two phases: order and disorder

$\mathbb{Z}_2$ Projective Transverse Field Ising Model

Substitute $Z_i Z_j \to \Pi^{ZZ}_{i,j}$, $X_i \to \Pi^{X}_{i}$; random time evolution, for $0 \leq p \leq 1$

- Measure X on each site w.p. p
- Measure ZZ on neighboring sites w.p. $1-p$
- Phase transitions in entanglement

(Assume periodic boundary conditions and 1D)



Lang and Büchler, Physical Review B, 2020

Quantifying Entanglement

Two main measures of entanglement in a system $\{s_1, \ldots, s_n\}$:

Definition

Entanglement Entropy: Measure of quantum entanglement between complementary subsystems of a bipartite state

$$S(\rho_A) = -\operatorname{Tr}(\rho_A \log_2 \rho_A)$$

Let $A = \{s_1, \ldots, s_{n/2}\}$.

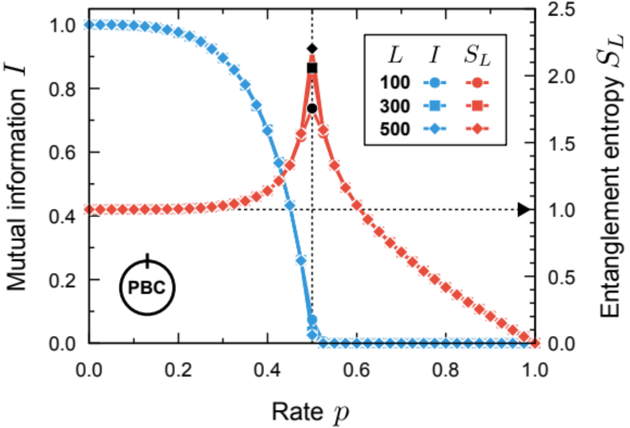
Definition

Mutual Information: Measure of correlation between two subsystems of a quantum state

$$I(A; B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB})$$

Let $A = s_1, B = s_{n/2}$.

$\mathbb{Z}_2$ Entanglement Transition



Lang and Büchler, Physical Review B, 2020

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Questions

- How can this model be generalized to higher (composite) dimensional qudits (namely $\mathbb{Z}_4$)?
- What entanglement-based phase transitions occur in higher-dimensional systems?

Generalizing to $\mathbb{Z}_4$ Model

Define Pauli $\mathcal{X}, \mathcal{Z}$ for four-state qudits as follows:

$$\mathcal{X} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\mathcal{Z} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

- $X \to \mathcal{X}, ZZ \to \mathcal{Z}\mathcal{Z}^\dagger$
- Introduce third competing measurement: $\mathcal{X}^2, \mathcal{Z}^2\mathcal{Z}^2$

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Efficient Classical Simulation of Qudit Chain

Need to simulate systems with $n \sim 100$ qudits

- Density matrix simulation – matrix dim scales as $2^n \times 2^n$
  - Complexity: $\mathcal{O}(\text{poly}(2^n))$
- Clifford simulation – need to perform Gaussian elimination on $n \times n$ matrix.
  - Complexity: $\mathcal{O}(n^3)$
- **Cluster model – update rule iterates over each site**
  - **Complexity:** $\mathcal{O}(n)$

$\mathbb{Z}_4$ Cluster Model

Keep tracks of two different types of clusters:

1. $\mathbb{Z}_2$ clusters – Formed using either $\mathcal{Z}^2\mathcal{Z}^2$ or $\mathcal{Z}\mathcal{Z}^\dagger$ and can be destroyed with $\mathcal{X}$ measurements.
2. $\mathbb{Z}_4$ clusters – Can only be formed using $\mathcal{Z}\mathcal{Z}^\dagger$ and can be destroyed with either $\mathcal{X}$ or $\mathcal{X}^2$ measurements.

Can represent state by storing two mappings (one for $\mathbb{Z}_2$ and $\mathbb{Z}_4$) from qudit sites to "colors" corresponding to which cluster each site is part of

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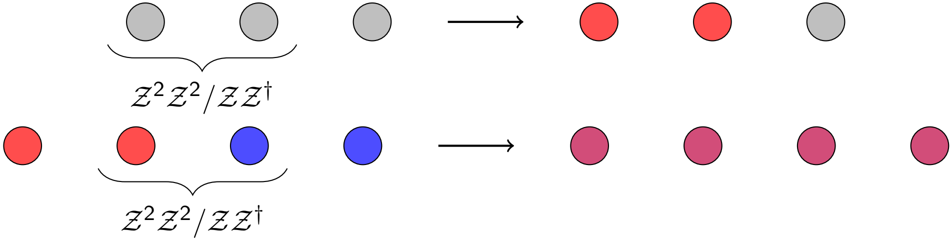
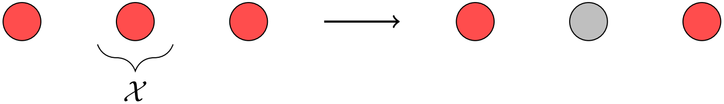
$\mathbb{Z}_2$ Clusters

$\mathcal{Z}^2\mathcal{Z}^2$ and $\mathcal{Z}\mathcal{Z}^\dagger$ measurements create/merge $\mathbb{Z}_2$ clusters:



$\mathcal{X}$ removes a site from a $\mathbb{Z}_2$ cluster



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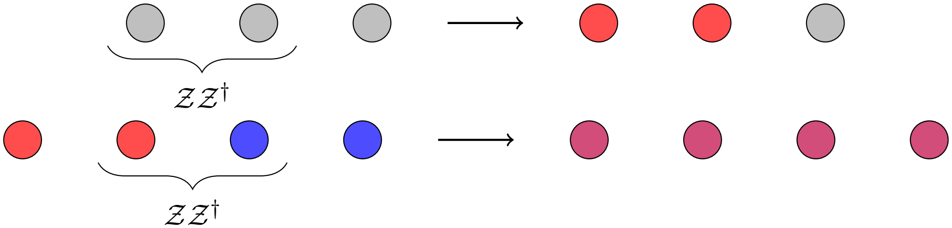

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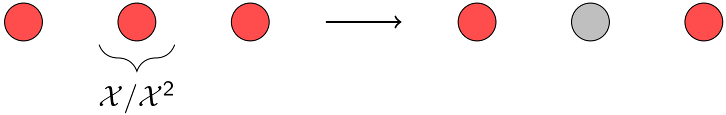

Entanglement Measurement

Values must be sample-averaged over thousands of trajectories

- Entanglement Entropy $S(\{s_1, \ldots, s_{n/2}\})$
  1. Partition qudit chain into two halves
  2. Count $\mathbb{Z}_2$ and $\mathbb{Z}_4$ clusters which cross the cut
- Mutual Information $I(s_1; s_{n/2})$
  1. For each cluster type, check if the two qudits are in the same cluster
  2. If they are in different clusters, do nothing to the mutual information.
  3. If there are no other qudits in the same cluster, add 2 to the total information, otherwise add 1.

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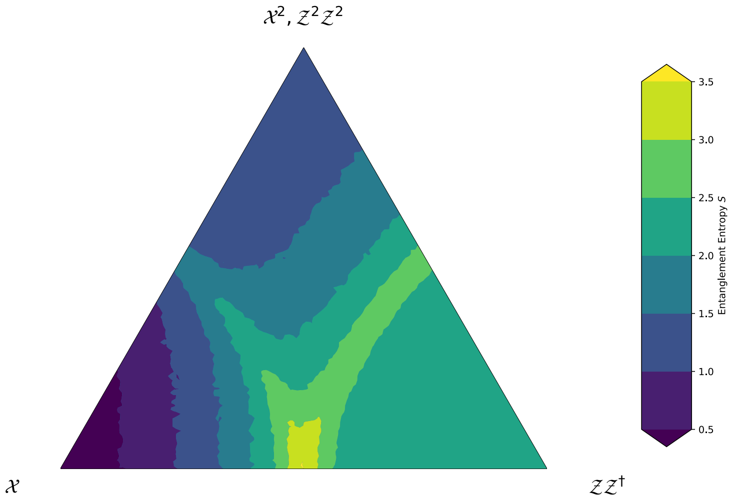
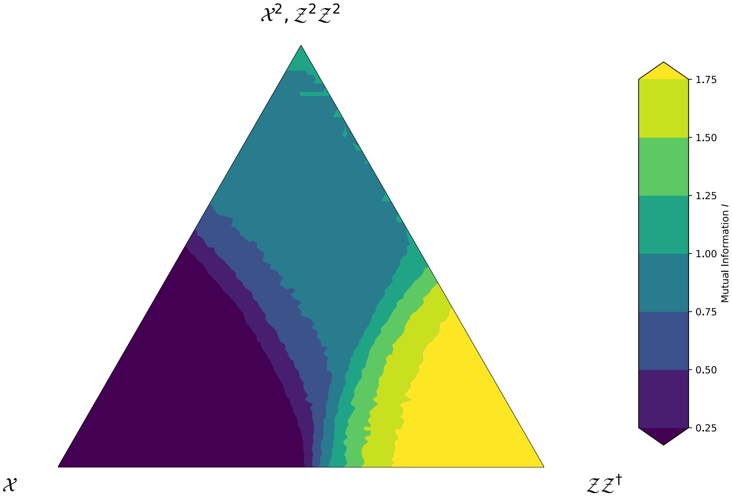
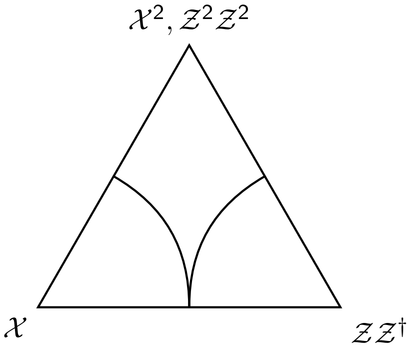
Entanglement Entropy

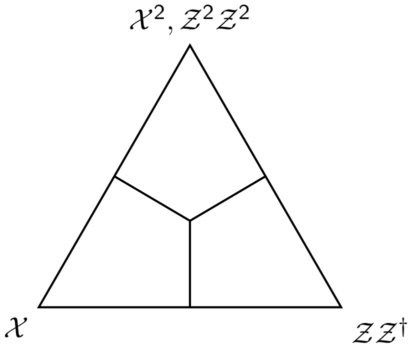


Mutual Information



Phase Diagram Comparison





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Coupled $\mathbb{Z}_2 \times \mathbb{Z}_2$ PTIM

Can the $\mathbb{Z}_4$ entanglement transitions be modeled using two coupled $\mathbb{Z}_2$ chains?

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Instead of a chain of 4-state qudits, consider two coupled chains of qubits:

$$\mathcal{X} \to X^{(1)}, X^{(2)}$$

$$\mathcal{Z}\mathcal{Z}^\dagger \to Z^{(1)}Z^{(1)}, Z^{(2)}Z^{(2)}$$

$$\mathcal{X}^2, \mathcal{Z}^2\mathcal{Z}^2 \to X^{(1)}X^{(2)}, Z^{(1)}Z^{(1)}Z^{(2)}Z^{(2)}$$

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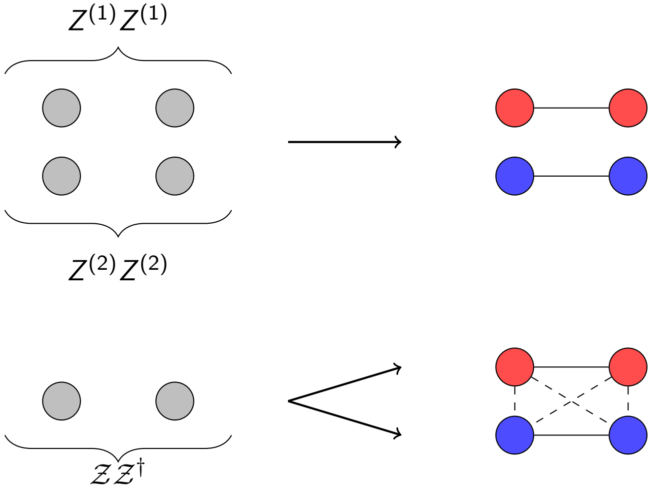
Are these two models completely equivalent in terms of entanglement behavior?

Interlayer Entropy

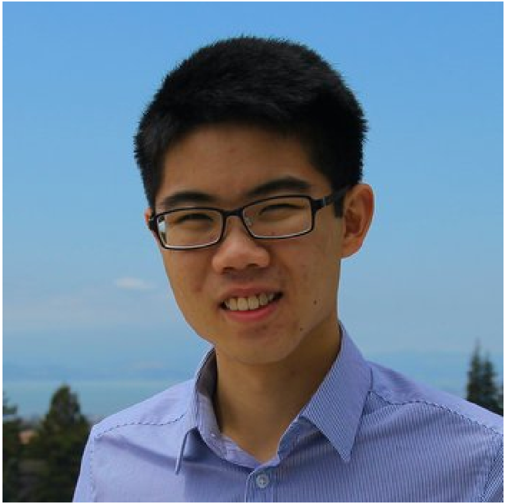


Conclusions

Altogether, we conclude the following:

- $\mathbb{Z}_4$ has three phases
- $\mathbb{Z}_4$ has different symmetry from equivalent clock model
- $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ have equivalent entanglement transitions
- $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ differ in interlayer entropy

Acknowledgements



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